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A NOTE ON TOTAL DOMINATION

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For a graph G, let $\gamma(G)$, $\gamma_t(G)$, $i(G)$ and $ir(G)$ denote the domination, total domination, independent domination and irredundance numbers of G, respectively. The following conjectures due to Robyn Dawes are proved:
(i) $\gamma(G)+\gamma_t(G)\leq p$ and
(ii) $i(G)+\gamma_t(G)\leq p$, where $|V(G)|=p>2$.
It is also shown that
(iii) $\gamma_t(G)\leq 2ir(G)$ and
(iv) $\gamma(G)\leq 2ir(G)-(k+1)$.
where k is the maximum number of isolates in an $ir(G)$ set. This last result improves the result, obtained independently by Bollóbas and Cockayne [6], Allan and Laskar [2].

1. Introduction

A vertex in a graph $G=(V, E)$ is said to *dominate* every vertex adjacent to it. A set D of vertices in G is a *dominating set* if every vertex in $V-D$ is dominated by at least one vertex in D. Dominating sets were defined by Berge [4] (where they are called *externally stable sets*) and Ore [16] and have been receiving increased attention recently. For a survey of results on dominating sets see [7, 15].

A well-known problem involving dominating sets (often called the Five Queens Problem) is to determine the smallest number of queens which can be placed on a chessboard so that every square is dominated by at least one queen. This problem, which dates at least back to 1892 (cf. Ball [3]), and the related Eight Queens Problem have been examined independently by many people, including Schuh [17], Kraitchik [14], Ball [3], de Jaenisch [12], König [13], Ahrens [1], Dudenay [10] and more recently Berge [5]. Among the many solutions to this problem, the two in Fig. 1 are particularly interesting. In the first solution (Fig. 1a) no queen is dominated by any other queen, while in the second solution (Fig. 1b) the opposite is essentially true, every queen is dominated by at least one other queen. The second solution suggests the following definition: a set T of vertices in G is a *total dominating set* if every vertex in V is dominated by at least one vertex in T. Total dominating sets were first defined and studied by Cockayne, Dawes and Hedetniemi [8]. In addition to several new results involving total domination, this note contains several new inequalities for the domination number of a graph.

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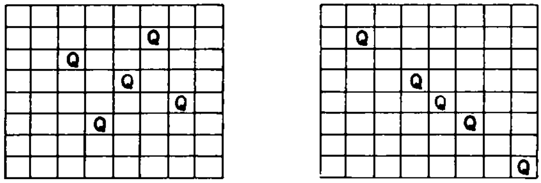


a b

Fig. 1

2. Definitions

We consider graphs $G=(V, E)$ which are finite, undirected, loopless and have no multiple edges. Any definitions not given here can be found in [5].

Define the (open) *neighborhood* $N(v)$ of a vertex to be the set of vertices adjacent to v, i.e. $N(v)=\{w \mid (v, w)\in E\}$; note that $v\notin N(v)$. The *closed neighborhood* of v is the set $N[v]=N(v)\cup\{v\}$. A set $D\subseteq V$ is a *dominating set* if $\forall v\in V-D$, $N(v)\cap D\neq\emptyset$. A set $T\subseteq V$ is a *total dominating set* if $\forall v\in V$, $N(v)\cap T\neq\emptyset$. A set $S\subseteq V$ is an *independent set* if no two vertices in S are adjacent; while a set $F\subseteq E$ of edges is *independent* if no two edges in F have a vertex in common. A set $S\subseteq V$ is *irredundant* if for every $v\in S$, $N[v]\not\subseteq \bigcup_{w\in S-v} N[w]$.

In what follows we will consider relationships among the following parameters of a graph G:

$ir(G)$, the *irredundance number of* G, is the minimum number of vertices in a maximal irredundant set of G.

$\gamma(G)$, the *domination number of* G, is the minimum number of vertices in a dominating set of G.

$\gamma_t(G)$, the *total domination number of* G, is the minimum number of vertices in a total dominating set of G.

$i(G)$, the *independent domination number of* G, is the minimum number of vertices in an independent dominating set (equivalently, in a maximal independent set) of G.

$\beta_0(G)$, the *independence number of* G, is the maximum number of vertices in an independent set of G.

$\alpha_0(G)$, the *vertex covering number of* G, is the minimum number of vertices in a set S such that every edge has at least one vertex in S.

$\beta_1(G)$, the *matching number of* G, is the maximum number of edges in an independent set.

$\beta_1^-(G)$, the *minimum matching number of* G, is the minimum number of edges in a maximal independent set.

In the following discussion we will abbreviate these parameters as $i=i(G)$, $\beta_0=\beta_0(G)$, $\gamma=\gamma(G)$, etc. We will also denote a set S as a γ-set, if S is a dominating set with $|S|=\gamma$. Similarly ir-set, β_0-set, etc. will be denoted.

3. Total domination

In the only paper published to date on the subject, Cockayne, Dawes and Hedetniemi [8] defined the total domination number and showed among other things that:

(i) if G is a connected graph with p vertices, then

$$\gamma_t \leqslant \tfrac{2}{3}p;$$

(ii) if G has p vertices, maximum degree $\Delta = \Delta(G)$, and no isolates, then

$$\gamma_t \leqslant p - \Delta + 1;$$

(iii) if G is connected and $\Delta < p-1$, then $\gamma_t \leqslant p - \Delta$;

(iv) if G has p vertices, no isolates and $\Delta < p-1$, then $\gamma_t + \bar{\gamma}_t \leqslant p+2$, with equality if and only if G or $\bar{G} = mK_2$, where $\bar{\gamma}_t = \gamma_t(\bar{G})$, and $\bar{G}$ denotes the complement of G; and

(v) if G has p vertices and no isolates, then

$$i \leqslant p + 1 - [(p-\gamma_t)/\gamma_t] - \tfrac{1}{2}\gamma_t.$$

Thus Cockayne, Dawes and Hedetniemi [8] related the total domination number, γ_t, to the number of vertices p, the maximum degree Δ and the independent domination number, i, of a graph G. In this paper we related γ_t to γ, i, β_1^- and ir. We also establish a new inequality between γ and ir.

4. New bounds for total domination

The following results are known:

$$\alpha_0 + \beta_0 = p, \qquad [11] \quad (1)$$

$$ir \leqslant \gamma \leqslant i \leqslant \beta_0, \qquad [7] \quad (2)$$

$$\gamma \leqslant \alpha_0. \qquad (3)$$

It is easy to establish the following:

$$\gamma \leqslant \gamma_t \leqslant 2\gamma, \qquad (4)$$

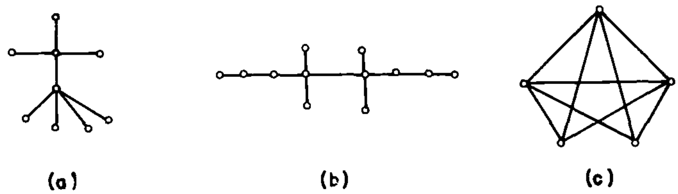
and

$$\text{if } \gamma_t = 2\gamma, \quad \text{then } \gamma = i. \qquad (5)$$

In view of (2), (3) and (4) it is natural to ask: What relationships, if any, exist between i, α_0 and γ_t? The graphs in Fig. 2 show that no particular inequality holds between any pair of these parameters. However, the following proposition follows immediately from (1), (2) and (3).

Proposition 1. *For any graph G without isolates,*

$$\gamma + i \leqslant p. \qquad (6)$$



	γ	α_0	i	γ_t
(a)	2	3	5	4
(b)	4	4	5	6
(c)	1	4	1	2

Fig. 2

On the basis of (6), Robyn Dawes [9] proposed the following

Conjecture. For any connected graph $G=(V,E)$, with $|V|>2$

(i) $\gamma+\gamma_t \leqslant p$, (7)

and

(ii) $i+\gamma_t \leqslant p$ (8)

Note that (7) might hold even if (8) does not; however, if (8) holds then (7) holds. We present a proof of (8).

Theorem 1. *If $G=(V,E)$ is a graph with $|V|=p$ such that each component has at least 3 vertices, then $i+\gamma_t \leqslant p$.*

Proof. Select from all γ-sets a γ-set V' which has a minimum number, say k, of isolated vertices in its induced subgraph $\langle V'\rangle$. Let V'' denote the isolated vertices of $\langle V'\rangle$. Note that $|V''|=k$. Now if $V''=\emptyset$, then $\gamma=\gamma_t$ and further, since $i+\gamma\leqslant p$ by (6), we have our result.

Hence, without loss of generality assume that $V''\neq\emptyset$. We may also assume that $\deg v\geqslant 2$ for each $v\in V''$, since if $\deg v=1$ for some $v\in V''$, then we can exchange it for the one vertex adjacent to v which by hypothesis must have degree $\geqslant 2$. (The new V' obtained by this exchange would still be a dominating set of cardinality γ and hence the vertex that v was exchanged for would still be isolated in the subgraph induced by the new V'; otherwise, we would contradict the minimality of k).

We claim that for each $v\in V'$, there exists a $w_v\in V-V'$ such that, $N(w_v)\cap V'=\{v\}$. If $v\in V'-V''$, since $N(v)\cap V'\neq\emptyset$, and V' is a minimum dominating set, such a w_v is guaranteed. On the other hand, if $v\in V''$, we know there exists a $w\in V-V'$, such that $N(w)\cap V'$ contains v. If for all such w,

$N(w) \cap V'$ contains another vertex $v' \neq v$, $v' \in V'$ then take a new v to be any vertex adjacent to v which contradicts the minimality of V''. Let $W = \bigcup_{v \in V'} \{w_v\}$ and note that $|W| = |V'| = \gamma$. Also, for each $v \in V''$, $|N(v)| \geqslant 2$ and $N(v) \cap V' = \emptyset$. Therefore, for each $v \in V''$ we pick $u_v \in N(v) - W$ and let $U = \bigcup_{v \in V''} \{u_v\}$. Note that $|U| = l \leqslant k$, $U \cap W = \emptyset$ and $U \cap V' = \emptyset$. Also $V' \cup U$ is a total dominating set, hence $\gamma_t \leqslant |V' \cup U| = \gamma + l$. Now extend V'' to a maximal independent set of G, denoted I. Notice that $i \leqslant |I|$ and that for each $v \in V'$, both v and w_v cannot be contained in I, since $vw_v \in E$. Therefore, it follows that there exists a set $W' \subset (W \cup V')$ such that $|W'| = \gamma$ and $W' \cap I = \emptyset$. Also $U \cap I = \emptyset$ and hence

$$\begin{aligned} i + \gamma_t &\leqslant |I| + \gamma + l \\ &= |I| + |W'| + |U| \\ &= |I \cup W' \cup U| \leqslant |V| = p. \end{aligned}$$

The next result improves the inequality $\gamma_t \leqslant 2\gamma$ given in (4).

Theorem 2. *For any connected graph G, $\gamma_t \leqslant 2ir$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_m\}$ be an *ir*-set. Since S is an irredundant set,

$$N[v_i] \subsetneq \bigcup_{j \neq i} N[v_j] \quad \forall v_i \in S.$$

Let

$$N_i = N[v_i] - \bigcup_{j \neq i} N[v_j] \quad \text{for } i = 1, 2, \ldots, m.$$

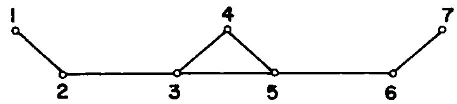
Since $N_i \neq \emptyset$, let $u_i \in N_i$ and let $S' = S \cup \{u_1, u_2, \ldots, u_n\}$. Note that $|S'| \leqslant 2ir$. We claim that S' is a dominating set. Suppose not, then let $w \in V - S'$ and $N(w) \cap S' = \emptyset$. Thus in particular

$$N[w] \not\subset \bigcup_{v_i \in S} N[v_i] \tag{9}$$

Consider $S \cup \{w\} = S_1$. Let $x \in S_1$. If $x = w$ then by (9), $N[w] \not\subset \bigcup_{v_i \in S} N[v_i]$. If $x = v_i \in S$, then there exists u_i such that, $u_i \in N_i$, and since $w \in N[u_i]$, for all i, we have $u_i \in N[v_i] - \bigcup_{\substack{v \in S_1 \\ v \neq v_i}} N[v]$. Therefore, for all $v_i \in S$,

$$N[v_i] \not\subset \bigcup_{v \in S_1 - u_i} N[v]. \tag{10}$$

Thus $S \cup \{w\} = S_1$ is an irredundant set. But this contradicts the maximality of S. Therefore S' is a dominating set with cardinality $\leqslant 2ir$. If v_i is an isolate in S, then select $u_i \in N_i$ such that $u_i \neq v_i$, such a u_i exists as G is connected. Thus, a total dominating set S'' can be constructed such that $\langle S'' \rangle$ does not contain any isolates, and consequently $\gamma_t \leqslant 2ir$.



(3,5) is an ir-set and (2,3,5,6) is an γ_t-set

Fig. 3

The bound in Theorem 2 is best possible as shown in the graph of Fig. 3. Our final bound relates γ_t with minimum maximal matchings.

Proposition 2. *For any connected graph G with $|v| \geqslant 2$ vertices, $\gamma_t \geqslant 2\beta_1^-$.*

Proof. Let M be a set of β_1^- edges that form a minimum maximal matching. Let V_M be the set of vertices in M. Then V_M is a total dominating set, since every vertex not in V_M must be adjacent to a vertex in V_M and every vertex in V_M is adjacent to another vertex in V_M. Since V_M has $2\beta_1^-$ vertices, the inequality is immediate.

5. A new bound for the domination number

The following inequality was independently obtained by Bollobas and Cockayne [6], Allan and Laskar [2]:

$$\gamma \leqslant 2ir - 1. \tag{11}$$

Using an argument very similar to the one used to prove Theorem 2, we can now improve this inequality.

Theorem 3. *Let S be an ir-set, and let the subgraph $\langle S\rangle$ induced by S have k isolated vertices. Then $\gamma(G) \leqslant 2ir(G) - (k+1)$.*

Proof. Let $S = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_m\}$ be an ir-set with k isolates $\{v_1, v_2, \ldots, v_k\}$. As before, let

$$N_i = N[v_i] - \bigcup_{j \neq i} N[\nu_j] \neq \emptyset.$$

Let $u_i \in N_i$ for $i = 1, 2, \ldots, m$, where $u_i = \nu_i$ for $i = 1, 2, \ldots, k$. Consider the set $S' = \{\nu_1, v_2, \ldots, \nu_{k+1}, \nu_{k+2}, \ldots, \nu_m, u_{k+1}, \ldots, u_m\}$. As in the proof of Theorem 2, S' is a dominating set and $|S'| = 2ir - k$. But S' is not a minimum dominating set, since it properly contains a maximal irredundant set. Therefore, $\gamma(G) \leqslant 2ir - (k+1)$.

Note. If S is an independent set, then S is an irredundant set.

Proposition 4. *If S is an ir-set, and S is independent then $ir = \gamma = i$.*

Proof. Clearly S is a dominating set; therefore, $\gamma \leqslant ir$, and $\gamma = i$. But $ir \leqslant \gamma$ by (1). Hence, $ir = \gamma = i$.

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